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The Aharonov–Anandan phase for quasi-energy trajectory-coherent states

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Abstract. Quasi-energy spectral series $[\varepsilon_\nu(\hbar), \Psi_{\varepsilon_\nu}]$ which, in the limit $\hbar \to 0$, correspond to stable motions of a classical system along closed phase trajectories are built up in terms of a quasi-classical approximation for the Schrödinger equation with an arbitrary T-periodic $\hbar^{-1}$-(pseudo)differential Hamilton operator. Using the procedure of splitting the quantum-mechanical phase into dynamic and geometric components, the 'geometric' contribution of the Aharonov–Anandan phase γ_{ε_ν} to the quasi-energy spectrum is calculated. It is shown that the γ_{ε_ν} phase, in the adiabatic approximation, coincides with the Berry phase that corresponds to a cyclic evolution of a stable rest-point of a classical system. Some examples are considered.

1. Statement of the problem

Let us consider a quantum-mechanical system whose Hamiltonian is described by an arbitrary Weyl-ordered $\hbar^{-1}$-(pseudo)differential scalar operator ($\hbar^{-1}$-PDO) $\overset{\omega}{\hat{H}}(t) = H(-i\hbar\frac{\partial}{\partial q}, q, t, \hbar)$, $q \in \mathbb{R}^n_q$, and, moreover, is a T-periodic function of time: $\overset{\omega}{\hat{H}}(t+T) = \overset{\omega}{\hat{H}}(t)$. The wavefunction of such a system satisfies the Schrödinger equation

$$\left(-i\hbar\partial_t + \overset{\omega}{\hat{H}}(t)\right)\Psi(q,t,\hbar) = 0. \tag{1.1}$$

Zel'dovich [1] and Ritus [2] were the first to distinguish an important class of solutions to equation (1.1), the quasi-energy states $\Psi_\varepsilon(q,t,\hbar)$, which can be presented in the form†

$$\Psi_\varepsilon(q,t,\hbar) = e^{(i/\hbar)\varepsilon t}\varphi_\varepsilon(q,t,\hbar) \tag{1.2}$$

where

$$\varphi_\varepsilon(q,t+T,\hbar) = \varphi_\varepsilon(q,t,\hbar). \tag{1.3}$$

The quantity ε entering into equation (1.2) namely a quasi-energy, is defined modulo $\hbar\omega$, ($\omega = 2\pi/T$), i.e. $\varepsilon' = \varepsilon + m\hbar\omega$, $m \in \mathbb{Z}$. States of this type play a key part in describing quantum-mechanical systems subjected to strong periodic external actions, when standard methods of the non-stationary perturbation theory appear to be inapplicable [7]. The results obtained with the use of the quasi-energy method are reviewed, for instance, in [8].

† The existence of a Floquet solution of the type (1.2) is an intricate mathematical problem. Substantial progress pertaining to this question has recently been made [3–6].

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Incidentally, as noted in [9], the quasi-energy states, equation (1.2), are a particular case of the cyclic states introduced by Aharonov and Anandan [10, 11] (see also [12]). Cyclic evolution of a quantum system on a time interval $[0, T]$ means that the state vector $\Psi(t)$ has the form

$$\Psi(t) = \mathrm{e}^{\mathrm{i}f(t)}\varphi(t) \qquad t \in [0, T] \tag{1.4}$$

where

$$f(T) - f(0) = \phi (\operatorname{mod} 2\pi) \tag{1.5}$$

$$\varphi(T) = \varphi(0). \tag{1.6}$$

The full phase ϕ of the wavefunction equation (1.4) is the sum of the dynamical phase

$$\delta = -\hbar^{-1} \int_0^T \mathrm{d}t \frac{\langle \Psi(t) | \overset{\hat{\omega}}{H} | \Psi(t) \rangle}{\langle \Psi(t) | \Psi(t) \rangle} \tag{1.7}$$

and the Aharonov–Anandan geometric phase (non-adiabatic Berry phase)

$$\gamma = \mathrm{i} \int_0^T \mathrm{d}t \frac{\langle \varphi(t) | \dot{\varphi}(t) \rangle}{\langle \varphi(t) | \varphi(t) \rangle}. \tag{1.8}$$

Here and below, the top dot denotes a time derivative. Comparing equations (1.2) and (1.4), we obtain that the function $f(t)$ for the case of quasi-energy states is given by

$$f(t) = -\hbar^{-1}\varepsilon t \tag{1.9}$$

and for the full phase ϕ, according to equation (1.5), we have

$$\phi = -\hbar^{-1}\varepsilon T (\operatorname{mod} 2\pi). \tag{1.10}$$

In view of equations (1.7)–(1.10), the Aharonov–Anandan phase γ_ε corresponding to a given quasi-energy state $\Psi_\varepsilon(q, t, \hbar)$ can be determined by the formula

$$\gamma_\varepsilon = -\hbar^{-1}\varepsilon T (\operatorname{mod} 2\pi) + \hbar^{-1} \int_0^T \mathrm{d}t \frac{\langle \Psi_\varepsilon | \overset{\hat{\omega}}{H}(t) | \Psi_\varepsilon \rangle}{\langle \Psi_\varepsilon \mid \Psi_\varepsilon \rangle}. \tag{1.11}$$

Note [13] that the theoretical status and the physical interpretation of a phase in quantum mechanics have not been fully established. The possibility of measuring it in experiments is essential (see [13–15] and references therein). Hence, the elaboration of effective approximate methods is an urgent problem.

Among the solutions to equation (1.1) that satisfy the quasi-peridocity condition, equation (1.2), one can distinguish a family of quasi-classical asymptotics Ψ_{ε_ν} having the following properties.

(i) The $\Psi_{\varepsilon_\nu}(q, t, \hbar)$ functions are mod $\mathrm{O}(\hbar^{5/2})$ approximation solutions to equation (1.1). This means that

$$\begin{gathered} \left(-\mathrm{i}\hbar\partial_t + \overset{\hat{\omega}}{H}(t)\right) \Psi_{\varepsilon_\nu}(q, t, \hbar) = v_\nu(q, t, \hbar) \\ \max_{t \in [0,T]} \| v_\nu(q, t, \hbar) \|_{L_2(\mathbb{R}^n_q)} = \mathrm{O}(\hbar^{5/2}). \end{gathered} \tag{1.12}$$

(ii) The solution Ψ_{ε_ν} approximates the corresponding exact solution Ψ to the Cauchy problem $\Psi|_{t=0} = \Psi_{\varepsilon_\nu}(q, 0, \hbar)$ to $\mathrm{O}(\hbar^{3/2})$ accuracy. In other words, on the time interval $[0, T]$ the estimate

$$\| \Psi(q, t, \hbar) - \Psi_{\varepsilon_\nu}(q, t, \hbar) \|_{L_2(\mathbb{R}^n_q)} = \mathrm{O}(\hbar^{3/2}) \tag{1.13}$$

is valid.

(iii) For every $t \in [0, T]$, the solutions, equation (1.12), form a complete orthonormal set with an accuracy of $O(\hbar^{3/2})$ in the space of states of the quantum system equation (1.1):

$$\langle \Psi_{\varepsilon_{\nu'}} \mid \Psi_{\varepsilon_\nu} \rangle = \delta_{\nu\nu'} + O(\hbar^{3/2}). \tag{1.14}$$

(iv) The functions Ψ_{ε_ν} have the form of wave packets localized in the neighbourhood of a given T-periodical classical trajectory. Such states, if any, are named quasi-energy trajectory-coherent states (TCSs).

The solutions Ψ_{ε_ν} and the respective quasi-energy ε_ν are built up in an explicit form in section 2. The Aharonov–Anandan phase γ_{ε_ν} for these states is calculated in the quasi-classical approximation (with an accuracy to $O(\hbar^{1/2})$) with the use of equation (1.11) in section 3. It should be particularly emphasized that the above-mentioned accuracy of approximation for the phase γ_{ε_ν} dictates the necessity of using Ψ_{ε_ν} states that satisfy the starting equation (1.1) with an accuracy of no less than $O(\hbar^{5/2})$†. Section 4 considers the case where the Hamiltonian of a quantum system depends on time t through a set of slowly varying T-periodic functions $R(t) = \{R_j(t)\}$, $j = \overline{1, N}$. An asymptotic expansion of the quantity γ_{ε_ν} in terms of the adiabaticity parameter T^{-1} is obtained. It is shown that the Aharonov–Anandan phase γ_{ε_ν} coincides, to a zero approximation, with the Berry phase [18] found in [19]. Some examples of the Aharonov–Andandan phase γ_{ε_ν} for quantum systems such as the sinusoidally forced harmonic oscillator and an electron in the Redmont field are given in section 5.

2. Construction of quasi-energy spectral series of the Schrödinger operator that conform to stable cycle motions of a classical system

2.1. The leading term of the asymptotic

The leading term $\overset{0}{\Psi}_{\varepsilon_\nu}$ of a quasi-classical TCS is an asymptotic $\mathrm{mod}\, O(\hbar^{3/2})$ solution to equation (1.1). It is built up in terms of Maslov's complex germ theory [20, 21]. Here, we describe briefly the algorithm for constructing such solutions.

The Weyl symbol‡ $H(p, q, t)$ of the operator $\overset{\omega}{\hat{H}}(t)$ will be assumed to be a smooth function of all its arguments $(p, q) \in \mathbb{R}^{2n}_{p,q}$, $t \in \mathbb{R}^1$, increasing, together with its derivatives, at $|p| \to \infty$ and $|q| \to \infty$ no more rapidly than some polynomial in $|p|$ and $|q|$, and uniformly in t. Let us relate the function $H(p, q, t)$ and the classical Hamiltonian system

$$\dot{p}(t) = -H_q(p, q, t) \qquad \dot{q}(t) = H_p(p, q, t). \tag{2.1}$$

Let $r_t = (p(t), q(t))$ be some fixed closed phase trajectory of the system (2.1), with period T. Assume that the system, in variables that corresponds to the Hamiltonian $H(p, q, t)$ and the T-periodic phase trajectory r_t,

$$\begin{aligned} \dot{W}(t) &= -H_{qp}(t)W(t) - H_{qq}(t)Z(t) \\ \dot{Z}(t) &= H_{pp}(t)W(t) + H_{pq}(t)Z(t) \end{aligned} \tag{2.2}$$

admits a set of $n (n = \dim \mathbb{R}^n_q)$ complex, linearly independent Floquet solutions $a_k(t) = (W_k(t), Z_k(t))^{\mathrm{T}}$:

$$a_k(t + T) = \mathrm{e}^{\mathrm{i}\Omega_k T} a_k(t) \qquad \mathrm{Im}\, \Omega_k = 0 \tag{2.3}$$

† Note that a similar situation occurs in the theory of spontaneous radiation [16, 17].

‡ Here and below, we consider the case of $\hbar^{-1}$-PDO whose symbols are independent of the parameter $\hbar$.

that satisfy the conditions

$$\{a_k, a_l\} = 0 \qquad \{a_k, \overset{*}{a}_l\} = 2\mathrm{i}\delta_{kl} \qquad k, l = \overline{1, n}. \tag{2.4}$$

Here, brackets $\{.,.\}$ denote the antisymmetric scalar product and $*$ denotes complex conjugation. It should be stressed that in terms of the Floquet theory for the linear Hamiltonian systems with periodic coefficients [22], the conditions (2.3) mean the phase trajectory r_t is stable in a linear approximation. The $2n$-dimensional vectors $a_k(t)$, $\overset{*}{a}_k(t)$, $k = \overline{1, n}$, form a symplectic basis in $\mathbb{C}^{2n}_{W,Z}$, and the n-dimensional complex plane $r^n(r_t)$ spanned by the vectors $a_k(t)$ forms a complex germ on r_t [21]. From the vectors $W(t)$, $Z(t)$, $k = \overline{1, n}$, let us construct the square $n \times n$ matrices

$$B(t) = (W_1(t), \ldots, W_n(t)) \qquad C(t) = (Z_1(t), \ldots, Z_n(t)). \tag{2.5}$$

The matrix $C(t)$ is non-singular; thus the symmetric matrix $Q(t) = B(t)C^{-1}(t)$ with the positively defined imaginary part

$$\operatorname{Im} Q(t) = [C^+(t)]^{-1}C^{-1}(t) > 0 \tag{2.6}$$

is defined. Here C^+ denotes a matrix Hermitian conjugate to C.

Let us introduce a complex action

$$\begin{aligned} S(q, t) = \int_0^t \mathrm{d}t(\langle p(t), \dot{q}(t)\rangle - H(t)) + \langle p(t), \Delta q\rangle + \tfrac{1}{2}\langle \Delta q, Q(t)\Delta q\rangle \\ \Delta q = q - q(t) \end{aligned} \tag{2.7}$$

and define a function of the WKB type, with the phase given by equation (2.7), as

$$\overset{0}{\Psi}_{\varepsilon_0}(q, t, \hbar) = |0, t\rangle = \frac{N_0(\hbar)}{\sqrt{\det C(t)}} \exp\left\{\frac{\mathrm{i}}{\hbar} S(q, t)\right\} \tag{2.8}$$

where $N_0(\hbar) = (\pi\hbar)^{-n/4}$ is a normalization factor. Let us set up a correspondence between the vectors $a_k(t)$, $\overset{*}{a}_k(t)$, $k = \overline{1, n}$, and the creation and annihilation operators

$$\begin{aligned} \hat{a}_k^+(t) &= \frac{1}{\sqrt{2\hbar}}(\langle \overset{*}{Z}_k(t), \Delta\hat{p}\rangle - \langle \overset{*}{W}_k(t), \Delta\hat{q}\rangle) \\ \hat{a}_k(t) &= \frac{1}{\sqrt{2\hbar}}(\langle Z_k(t), \Delta\hat{p}\rangle - \langle W_k(t), \Delta\hat{q}\rangle) \end{aligned} \tag{2.9}$$

where $\Delta\hat{p} = -\mathrm{i}\hbar\partial_q - p(t)$. It can readily be ascertained that the relationships

$$[\hat{a}_k, \hat{a}_l] = [\hat{a}_k^+, \hat{a}_l^+] = 0 \qquad [\hat{a}_k, \hat{a}_l^+] = \delta_{kl} \tag{2.10}$$

$$\hat{a}_k|0, t\rangle = 0 \qquad l, k = \overline{1, n} \tag{2.11}$$

are valid. Acting sequentially with the creation operators $\hat{a}_k^+$ on the vacuum state (2.8), let us build a set of functions of the form

$$\overset{0}{\Psi}_{\varepsilon_\nu}(q, t, \hbar) = |\nu, t\rangle = \prod_{k=1}^{n} \frac{1}{\sqrt{\nu!}}(\hat{a}_k^+)^{\nu_k}|0, t\rangle. \tag{2.12}$$

Such states are called trajectory-coherent states (TCSs) in [23]. At every $t \in [0, T]$, it can be shown that the functions $|\nu, t\rangle$ form an orthonormal set, complete in $L_2(\mathbb{R}^n_q)$, of solutions to the Schrödinger equation:

$$(-\mathrm{i}\hbar\partial_t + \hat{H}_0(t))|\nu, t\rangle = 0 \tag{2.13}$$

$$\langle \nu', t \mid \nu, t\rangle = \delta_{\nu'\nu} \tag{2.14}$$

where $\hat{H}_0(t)$ is a square-law operator of the form

$$\hat{H}_0(t) = H(t) + \hat{\delta}^1 H(t) + \tfrac{1}{2}\hat{\delta}^2 H(t). \tag{2.15}$$

Here, the operator designation $\hat{\delta}^k H(t)$ refers to the kth term in the Taylor series expansion of the Weyl-ordered operator $\overset{\omega}{\hat{H}}(t)$ in terms of powers of the operators $\Delta\hat{p}$ and Δq in the neighbourhood of the phase curve r_t, i.e.

$$\hat{\delta}^k H(t) = \left(\left\langle \Delta\hat{p}, \frac{\partial}{\partial z}\right\rangle + \left\langle \Delta q, \frac{\partial}{\partial y}\right\rangle\right)^k H(z, y, t)\bigg|_{\substack{z=p(t)\\ y=q(t)}}. \tag{2.16}$$

Let us introduce a class of functions of the form

$$\mathcal{P}^t_\hbar = \left\{f, f = \exp\left[\frac{\mathrm{i}}{\hbar}(S(t) + \langle p(t), \Delta q\rangle)\right]\phi\left(\frac{\Delta q}{\sqrt{\hbar}}, t\right), \phi(\zeta, t) \in \mathbb{S}\right\}$$

$$S(t) = \int_0^t (\langle p(t), \dot{q}(t)\rangle - H(t))\,\mathrm{d}t \tag{2.17}$$

where $\phi(\zeta, t)$ is a smooth function in $t \in [0, T]$ and $\mathbb{S}$ is a Schwartz space with respect to $\zeta \in \mathbb{R}^n$. The explicit form of the functions (2.12), testifies to the fact that they form an orthogonal basis in the space $\mathcal{P}^t_\hbar$. Hence, it follows, in particular, that $\mathcal{P}^t_\hbar$ is dense throughout $L_2(\mathbb{R}^n_q)$. Let $\hat{\mathrm{O}}(\hbar^\alpha)$ designates an operator $\hat{F}$: $L_2(\mathbb{R}^n_q) \to L_2(\mathbb{R}^n_q)$ for which the estimation $\|\hat{F}\varphi\|_{L_2(\mathbb{R}^n_q)} = \mathrm{O}(\hbar^\alpha)$, $\alpha > 0$, is valid on the set $\varphi \in \mathcal{P}^t_\hbar$. It can easily be checked that in this sense, the asymptotic estimations

$$\Delta\hat{p} = \hat{\mathrm{O}}(\hbar^{1/2}) \qquad \Delta q = \hat{\mathrm{O}}(\hbar^{1/2}) \tag{2.18}$$

$$[-\mathrm{i}\hbar\partial_t + H(t) + \hat{a}_0(t)] = \hat{\mathrm{O}}(\hbar) \tag{2.19}$$

where $\hat{a}_0(t) = \hat{\delta}^1 H(t) = \langle \dot{q}(t), \Delta\hat{p}\rangle - \langle \dot{p}(t), \Delta q\rangle$, are valid (see e.g. [21, 24]).

Let us expand the operator $\overset{\omega}{\hat{H}}(t)$ in the Taylor power series over the operators $\Delta\hat{p}$ and Δq to the second order:

$$\overset{\omega}{\hat{H}}(t) = \hat{H}_0(t) + \hat{R}_3. \tag{2.20}$$

By virtue of the fact that for the remainder term $\hat{R}_3$ of the Taylor series (2.20), $\hat{R}_3 = \hat{\mathrm{O}}(\hbar^{3/2})$ is valid, we obtain, with the use of equations (2.13) and (2.19), that the functions (2.12) are approximate $\mathrm{mod}\,\mathrm{O}(\hbar^{3/2})$ solutions to equation (1.1).

Further consideration requires some issues from the preceding constructions. Thus, solving equation (2.9) for the operators $\Delta\hat{p}$ and Δq, we obtain

$$\Delta\hat{p} = \mathrm{i}\sqrt{\hbar/2}\left(\overset{*}{B}(t)\hat{a} - B(t)\hat{a}^+\right) \qquad \Delta q = \mathrm{i}\sqrt{\hbar/2}\left(\overset{*}{C}(t)\hat{a} - C(t)\hat{a}^+\right) \tag{2.21}$$

where $\hat{a}^+ = (\hat{a}^+_1, \ldots, \hat{a}^+_n)^{\mathrm{T}}$, $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_n)^{\mathrm{T}}$. Furthermore, with equation (2.4) taken into account, it can readily be checked that

$$\begin{aligned} &C^+(t)B(t) - B^+(t)C(t) = 2\mathrm{i}\mathbb{I} \qquad && C^{\mathrm{T}}(t)B(t) - B^{\mathrm{T}}(t)C(t) = 0 \\ &\overset{*}{C}(t)C^{\mathrm{T}}(t) - C(t)C^+(t) = 0 && \overset{*}{B}(t)B^{\mathrm{T}}(t) - B(t)B^+(t) = 0. \end{aligned} \tag{2.22}$$

Here, C^{T} denote matrices transposed to C, and $\mathbf{I} \doteq \|\delta_{ij}\|_{n\times n}$. Finally, using equations (2.10) and (2.11) we may prove the validity of the identities

$$\begin{aligned} &\langle\nu|\hat{a}_k|\nu\rangle = \langle\nu|\hat{a}^+_k|\nu\rangle = 0 \qquad && \langle\nu|\hat{a}_k\hat{a}_l|\nu\rangle = \langle\nu|\hat{a}^+_k\hat{a}^+_l|\nu\rangle = 0 \\ &\langle\nu|\hat{a}^+_k\hat{a}_l|\nu\rangle = \nu_k\delta_{kl} && \langle\nu|\hat{a}_k\hat{a}^+_l|\nu\rangle = (\nu_k + 1)\delta_{kl}. \end{aligned} \tag{2.23}$$

2.2. Construction of quasi-energy TCSs (mod $O(\hbar^{5/2})$)

The scheme for constructing the states Ψ_{ε_ν} that satisfy equations (1.12)–(1.14) is similar to the general scheme for constructing the high-order approximations for the TCSs of the Schrödinger equation [24]. Note that in the case of a time-independent Hamiltonian, the high-order approximations for the semiclassical wave packets were also constructed in [25–29].

We shall find the solution to the problem (1.12) in the form

$$\Psi_{\varepsilon_\nu} = |\nu, t\rangle + \sqrt{\hbar}\varphi_\nu^{(1)} + \hbar\varphi_\nu^{(2)} + O(\hbar^{3/2}) \tag{2.24}$$

where $\varphi_\nu^{(1)}$ and $\varphi_\nu^{(2)}$ are unknown functions from the class $\mathcal{P}_\hbar^t$. Let us expand the operator $\overset{\omega}{H}(t)$ in the Taylor power series over the operators $\Delta\hat{p}$ and Δq up to fourth order included, and substitute equation (2.24) into equation (1.1). Then, gathering together the terms of the same order in $\hbar^{1/2}$, we obtain, in consideration of equation (2.13), the set of equations

$$(-i\hbar\partial_t + \hat{H}_0(t))\sqrt{\hbar}\varphi_y^{(1)} + \hat{H}^{(3)}(t)|\nu, t\rangle = 0 \tag{2.25}$$

$$(-i\hbar\partial_t + \hat{H}_0(t))\hbar\varphi_\nu^{(2)} + \hat{H}^{(3)}(t)\sqrt{\hbar}\varphi_\nu^{(1)} + \hat{H}^{(4)}(t)|\nu, t\rangle = 0 \tag{2.26}$$

where

$$\hat{H}^{(j)}(t) = \frac{1}{(j)!}\hat{\delta}^j H(t) = \hat{O}(\hbar^{j/2}) \qquad j = 3, 4. \tag{2.27}$$

The functions $\varphi_\nu^{(j)}$, $j = 1, 2$, can be found in the form of an expansion in terms of the complete orthonormal set of states $|\nu, t\rangle$. Then, having determined the coefficients of the above-mentioned expansion, we find

$$\varphi_\nu^{(1)} = -i\hat{\mathcal{F}}_1(t)|\nu, t\rangle + \sum_{|\nu'|=0}^{\infty} C^1_{\nu\nu'}|\nu', t\rangle \tag{2.28}$$

$$\varphi_\nu^{(2)} = -i\hat{\mathcal{F}}_1(t)\varphi_\nu^{(1)} - i\hat{\mathcal{F}}_2(t)|\nu, t\rangle + \sum_{|\nu'|=0}^{\infty} C^2_{\nu\nu'}|\nu', t\rangle \tag{2.29}$$

where the operators $\hat{\mathcal{F}}_j(t)$, $j = 1, 2$, have the form

$$\hbar^{j/2}\hat{\mathcal{F}}_j(t)\varphi(t) = \frac{1}{\hbar(j+2)!}\sum_{|\nu'|=0}^{\infty}|\nu', t\rangle\int_0^t d\tau\langle\nu', \tau|\hat{\delta}^{(j+2)}H(\tau)|\varphi(\tau)\rangle \tag{2.30}$$

and the integration constants $C^1_{\nu\nu'}$, $C^2_{\nu\nu'}$ are chosen such that the conditions for quasi-periodicity and orthonormality of the states, equation (2.24), are fulfilled. Using equations (2.28) and (2.29) we find for the functions Ψ_{ε_ν}, the expansion

$$\Psi_{\varepsilon_\nu}(q, t, \hbar) = \left[1 - i\sqrt{\hbar}\hat{\mathcal{F}}_1(t) - i\hbar\hat{\mathcal{F}}_2(t) - \hbar\hat{\mathcal{F}}_1^2(t)\right]\sum_{|\nu'|=0}^{\infty} C_{\nu\nu'}(\hbar)|\nu', t\rangle + O(\hbar^{3/2}). \tag{2.31}$$

Here

$$C_{\nu\nu'}(\hbar) = \delta_{\nu\nu'} + \sum_{k=1}^{2}\hbar^{k/2}C^k_{\nu\nu'}. \tag{2.32}$$

To substantiate the validity of the estimate (1.13) we use the standard expedient as follows. Let $\hat{U}_\hbar(t)$ be the time evolution operator for system (1.1). Then, in accordance with the Duhamel principle we have

$$\Psi(q, t, \hbar) = \Psi_{\varepsilon_\nu}(q, t, \hbar) + \frac{1}{i\hbar}\int_0^t d\tau\,\hat{U}_\hbar(t-\tau)v_\nu(q, \tau, \hbar). \tag{2.33}$$

Hence, in view of the unitary nature of the operator $\hat{U}_\hbar(t)$, we obtain on the time interval $[0, T]$

$$\|\Psi - \Psi_{\varepsilon_\nu}\|_{L_2} \leqslant \hbar^{-1}\int_0^t \|\hat{U}_\hbar(t-\tau)\|\|v_\nu(q,\tau,\hbar)\|_{L_2}\,\mathrm{d}\tau \leqslant T/\hbar \max_{0\leqslant\tau\leqslant T}\|v_\nu(q,\tau,\hbar)\|_{L_2}$$
$$= \mathrm{O}(\hbar^{3/2}).$$

2.3. The quasi-energy spectral problem

To obtain a quasi-energy spectrum that corresponds, in the limit $\hbar \to 0$, to a stable motion of a classical system along a closed trajectory r_t, we isolate the functions from the family (2.31) that satisfy modulo $\mathrm{O}(\hbar^{3/2})$ the quasi-periodicity condition

$$\Psi_{\varepsilon_\nu}(q, t+T, \hbar) = \mathrm{e}^{-\mathrm{i}\hbar^{-1}\varepsilon_\nu T}\Psi_{\varepsilon_\nu}(q,t,\hbar) + \mathrm{O}(\hbar^{3/2}). \tag{2.34}$$

The quasi-energy ε_ν will be found in the form of the expansion in $\hbar$:

$$\varepsilon_\nu = \sum_{k=0}^{2}\hbar^k\varepsilon_\nu^{(k)} + \mathrm{O}(\hbar^{5/2}). \tag{2.35}$$

Substituting equations (2.24) and (2.35) into equation (2.34) and equating summands with the same power of $\hbar^{1/2}$, we obtain the chain of equalities

$$|\nu_3, t+T\rangle = \mathrm{e}^{-\mathrm{i}\beta_\nu T}|\nu, t\rangle \tag{2.36}$$

$$\varphi_\nu^{(1)}(t+T) = \mathrm{e}^{-\mathrm{i}\beta_\nu T}\varphi_\nu^{(1)}(t) \tag{2.37}$$

$$\varphi_\nu^{(2)}(t+T) = \mathrm{e}^{-\mathrm{i}\beta_\nu T}(\varphi_\nu^{(2)}(t) - \mathrm{i}\varepsilon_\nu^{(2)}(T)|\nu, t\rangle) \tag{2.38}$$

where $\beta_\nu = \hbar^{-1}\varepsilon_\nu^{(0)} + \varepsilon_\nu^{(1)}$. From equation (2.36), in view of the explicit form (2.12) of the functions $|\nu, \tau\rangle$ and relationships (2.3), it follows that

$$\beta_\nu(\mathrm{mod}\,\omega) = -\frac{1}{\hbar T}\int_0^T \mathrm{d}t(\langle p(t), \dot{q}(t)\rangle - H(t)) + \sum_{k=1}^{n}\Omega_k(\nu_k + \tfrac{1}{2}) \tag{2.39}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is a set of non-negative integers defined in equation (2.12). The conditions (2.37) and (2.38) in turn allow us to find the constants $C_{\nu\nu'}$, $\nu \neq \nu'$, and the quasi-energy additive $\varepsilon_\nu^{(2)}$ (see appendix A).

Finally, we must show that with properly chosen constants $C_{\nu\nu}(\hbar) = (1+\sum_{k=1}^{2}\hbar^{k/2}C_\nu^k)$, the functions (2.31) form an orthonormal ($\mathrm{mod}\,\mathrm{O}(\hbar^{3/2})$) set of states. To do this we use the identities†

$$\langle\nu|\hat{\mathcal{F}}_j^+|\nu'\rangle - \langle\nu|\hat{\mathcal{F}}_j|\nu'\rangle = 0 \tag{2.40}$$

that follow from the definition (2.30) of the operators $\hat{\mathcal{F}}_j$. Then it can readily be seen that

$$\langle\Psi_{\varepsilon_\nu} \mid \Psi_{\varepsilon_\nu}\rangle = \langle\nu|(1 + \hbar[\hat{\mathcal{F}}_1^+\hat{\mathcal{F}}_1 - \hat{\mathcal{F}}_1^2 - (\hat{\mathcal{F}}_1^+)^2])|\nu\rangle + 2\sqrt{\hbar}\,\mathrm{Re}\,\overset{1}{C}_\nu + 2\hbar\,\mathrm{Re}\,\overset{2}{C}_\nu$$
$$+\hbar\sum_{|\nu'|=0}^{\infty}|\overset{1}{C}_{\nu\nu'}|^2 + \mathrm{O}(\hbar^{3/2}). \tag{2.41}$$

Further, we put

$$\frac{1}{3!\hbar}\langle\nu|\hat{\delta}^3H(t)|\nu'\rangle = \Delta_{\nu'}(t) \qquad \frac{1}{3!\hbar}\langle\nu'|\hat{\delta}^3H(t)|\nu\rangle = \tilde{\Delta}_{\nu'}(t). \tag{2.42}$$

† To simplify designations, here we will ignore the time dependence of the functions $|\nu, t\rangle$ and the operators $\hat{\mathcal{F}}_j(t)$.

Then for the operators in the square brackets in equation (2.41) we find

$$\begin{aligned}
\langle\nu|\hbar\hat{\mathcal{F}}_1^{+}\hat{\mathcal{F}}_1|\nu\rangle &= \sum_{|\nu'|=0}^{\infty}\int_0^t \mathrm{d}\tau_1\int_0^t \mathrm{d}\tau_2\Delta_{\nu'}(\tau_1)\tilde{\Delta}_{\nu'}(\tau_2)\\
\langle\nu|\hbar\hat{\mathcal{F}}_1^{2}|\nu\rangle &= \sum_{|\nu'|=0}^{\infty}\int_0^t \mathrm{d}\tau_1\Delta_{\nu'}(\tau_1)\int_0^{\tau_1} \mathrm{d}\tau_2\tilde{\Delta}_{\nu'}(\tau_2)\\
\langle\nu|\hbar(\hat{\mathcal{F}}_1^{+})^2|\nu\rangle &= \sum_{|\nu'|=0}^{\infty}\int_0^t \mathrm{d}\tau_2\tilde{\Delta}_{\nu'}(\tau_2)\int_0^{\tau_2} \mathrm{d}\tau_1\Delta_{\nu'}(\tau_1).
\end{aligned} \tag{2.43}$$

In consideration of equation (2.43), expression (2.41) takes the form

$$\begin{aligned}
\langle\Psi_{\varepsilon_\nu}\mid\Psi_{\varepsilon_\nu}\rangle &= 1+2\sqrt{\hbar}\,\mathrm{Re}\,\overset{1}{C}_\nu+2\hbar\,\mathrm{Re}\,C_\nu^2+\hbar\sum_{|\nu'|=0}^{\infty}|\overset{1}{C}_{\nu\nu'}|^2+\int_0^t\mathrm{d}\tau_1\int_0^t\mathrm{d}\tau_2\Delta(\tau_1,\tau_2)\\
&\quad-\int_0^t\mathrm{d}\tau_1\int_0^{\tau_1}\mathrm{d}\tau_2\Delta(\tau_1,\tau_2)-\int_0^t\mathrm{d}\tau_2\int_0^{\tau_2}\mathrm{d}\tau_1\Delta(\tau_1,\tau_2)+\mathrm{O}(\hbar^{3/2})
\end{aligned} \tag{2.44}$$

where $\Delta(\tau_1,\tau_2) = \sum_{|\nu'|=0}^{\infty}\Delta_{\nu'}(\tau_1)\tilde{\Delta}_{\nu'}(\tau_2)$. The sum of the last three summands in equation (2.44) is identically zero. Hence, for

$$\mathrm{Re}\,\overset{1}{C}_\nu = 0 \qquad \mathrm{Re}\,\overset{2}{C}_\nu = -\frac{1}{2}\sum_{|\nu'|=0}^{\infty}|\overset{1}{C}_{\nu\nu'}|^2 \tag{2.45}$$

the right-hand side of equation (2.44) is equal modulo $\mathrm{O}(\hbar^{3/2})$ to unity. The property of orthogonality (mod $\mathrm{O}(\hbar^{3/2})$) of functions (2.31) for $\nu\neq\nu'$ is proved in a similar way.

From the above results it follows that the quasi-energy TCSs (2.12), in the important particular case of quantum systems with a periodic square-law operator, form a complete orthonormal set of exact solutions, and the quasi-energies corresponding to these states are $\varepsilon_\nu = \hbar\beta_\nu$, where β_ν are given by equation (2.39). Exact results related to the construction of quasi-energy states for square-law systems were also obtained by Malkin and Man'ko [8]. Moroever, it may be said that in [8, 30, 31] the quasi-energy spectral series for these systems is constructed in the region of instability motions. The mathematical procedure developed in [8] can turn out to be useful for generalization of the results of this section in the case of unstable isolated periodic orbits [32].

Finally, the works [33, 34] should be noted; they are devoted to the quasi-classical quantization of T-periodic systems. Here, as distinct from the standard approach, the time $t\,(\mathrm{mod}\,T)$ plays the part of an additional angular variable, and the motion of a classical system is considered in an extended phase space $\{q, t, p, p_t\}$. In this consideration, the quasi-energy spectral series are generated by the quantization conditions of these motions in accordance with the well known Einstein–Brillouin–Keller procedure (with Maslov's topological characteristics taken into account).

3. The Aharonov–Anandan phase

Now we are going to consider the calculation of the Aharonov–Anandan phase corresponding to the quasi-energy states given by equations (2.31) and (2.32). To do this, we use formula (1.11) and neglect the $\mathrm{O}(\hbar^{1/2})$ order summands:

$$\gamma_{\varepsilon_\nu} = -\beta_\nu T+\hbar^{-1}\int_0^T \mathrm{d}t\{H(t)+\hbar(\langle\dot{q},\overset{1}{P}_{(\nu)}(t)\rangle-\langle\dot{p},\overset{1}{q}_{(\nu)}(t)\rangle)+\tfrac{1}{2}\langle\nu|\hat{\delta}^2H(t)|\nu\rangle\}. \tag{3.1}$$

Here, the vectors $\overset{1}{p}_{(\nu)}(t)$ and $\overset{1}{q}_{(\nu)}(t)$ are defined by the formulae

$$\begin{aligned}\langle\Psi_{\varepsilon_\nu}(t)|\Delta\hat{p}|\Psi_{\varepsilon_\nu}(t)\rangle &= \hbar\,\overset{1}{p}_{(\nu)}(t) + \mathrm{O}(\hbar^{3/2})\\ \langle\Psi_{\varepsilon_\nu}(t)|\Delta q|\Psi_{\varepsilon_\nu}(t)\rangle &= \hbar\,\overset{1}{q}_{(\nu)}(t) + \mathrm{O}(\hbar^{3/2})\end{aligned} \tag{3.2}$$

and specify small ($\mathrm{O}(\hbar)$ order, $\hbar \to 0$) fluctuations of a quantum particle in the Ψ_{ε_ν} around a classical phase trajectory $r_t = (p(t), q(t))$. Using equations (2.21)–(2.23) we can then easily calculate the correlation matrix that characterizes the quantum fluctuations of the dynamic variables $\hat{p} = -\mathrm{i}\hbar\partial_q$, $\hat{q} = q$, relative to their average values $\langle\hat{p}\rangle$, $\langle\hat{q}\rangle$ in the state defined by the leading term $|\nu, t\rangle$ of the asymptotic given by equation (2.12):

$$\begin{aligned}(\sigma_{qq})_{kl} &= \langle\nu|\Delta q_k\Delta q_l|\nu\rangle = \frac{\hbar}{4}\Big(C(t)D(\nu)C^+(t) + \overset{*}{C}(t)D(\nu)C^{\mathrm{T}}(t)\Big)_{kl}\\ (\sigma_{pp})_{kl} &= \langle\nu|\Delta\hat{p}_k\Delta\hat{p}_l|\nu\rangle = \frac{\hbar}{4}\Big(B(t)D(\nu)B^+(t) + \overset{*}{B}(t)D(\nu)B^{\mathrm{T}}(t)\Big)_{kl}\\ (\sigma_{pq})_{kl} &= \tfrac{1}{2}\langle\nu|[\Delta\hat{p}_k\Delta q_l]_+|\nu\rangle = \frac{\hbar}{4}\Big(B(t)D(\nu)C^+(t) + \overset{*}{B}(t)D(\nu)C^{\mathrm{T}}(t)\Big)_{kl}.\end{aligned} \tag{3.3}$$

Here, $D(\nu) = \mathrm{diag}(2\nu_1 + 1, \ldots, 2\nu_n + 1)$, $\sigma_{AB} = \frac{1}{2}\langle[\hat{A}, \hat{B}]_+\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle = \frac{1}{2}\langle[\Delta\hat{A}, \Delta\hat{B}]_+\rangle$, where $\Delta\hat{A} = \hat{A} - \langle\hat{A}\rangle$, $\Delta\hat{B} = \hat{B} - \langle\hat{B}\rangle$, $[\hat{A}, \hat{B}]_+ = \hat{A}\hat{B} + \hat{B}\hat{A}$. In deriving equations (3.3) we used the equalities $\langle\nu|\hat{p}|\nu\rangle = p(t)$, $\langle\nu|\hat{q}|\nu\rangle = q(t)$. With equations (3.3), it can readily be shown that

$$\begin{aligned}\langle\nu|\hat{\delta}^2 H(t)|\nu\rangle &= \frac{\hbar}{2}\,\mathrm{Sp}\,\mathrm{Re}(\dot{C}(t)D(\nu)B^+(t) - \dot{B}(t)D(\nu)C^+(t))\\ &= -\hbar\,\mathrm{Re}\sum_{k=1}^{n}(\nu_k + \tfrac{1}{2})\Big\{\dot{a}_k(t), \overset{*}{a}_k(t)\Big\}.\end{aligned} \tag{3.4}$$

Note that the quantity under the summation sign on the right-hand side of equation (3.4) in view of equation (2.4), is real; hence, the Re sign can be omitted.

Let us introduce vectors $a_0(t) = (\dot{p}(t), \dot{q}(t))^{\mathrm{T}}$ and $\chi_{(\nu)}(t) = \Big(\overset{1}{p}_{(\nu)}(t), \overset{1}{q}_{(\nu)}(t)\Big)^{\mathrm{T}}$. Then, substituting the explicit expression for β_ν, equation (2.39), into equation (3.1), and using equation (3.4), we find

$$\begin{aligned}\gamma_{\varepsilon_\nu} = \hbar^{-1}\int_0^T \mathrm{d}t\,\langle p(t), \dot{q}(t)\rangle - \int_0^T \mathrm{d}t\{a_0(t), \chi_{(\nu)}(t)\}\\ -\sum_{k=1}^{n}(\nu_k + \tfrac{1}{2})\Big[T\Omega_k + \frac{1}{2}\int_0^T \mathrm{d}t\{\dot{a}_k(t), \overset{*}{a}_k(t)\}\Big].\end{aligned} \tag{3.5}$$

If instead we now introduce the vectors, equation (2.3), T-periodic vector-functions

$$\tilde{a}_k(t) = \mathrm{e}^{-\mathrm{i}\Omega_k t}a_k(t) \qquad \tilde{a}_k(t + T) = \tilde{a}_k(t) \tag{3.6}$$

equation (3.5) will take the form

$$\gamma_{\varepsilon_\nu} = \hbar^{-1}\int_0^T \mathrm{d}t\,\langle p(t), \dot{q}(t)\rangle - \frac{1}{2}\sum_{k=1}^{n}(\nu_k + \tfrac{1}{2})\int_0^T \mathrm{d}t\,\Big\{\dot{\tilde{a}}_k, \overset{*}{\tilde{a}}_k\Big\} - \int_0^T \mathrm{d}t\{a_0(t), \chi_{(\nu)}(t)\}. \tag{3.7}$$

Using the equations of motion for the quantum-mechanical average of the operators $\hat{p}$ and $\hat{q}$ in the state Ψ_{ε_ν}:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\Psi_{\varepsilon_\nu}|\hat{p}|\Psi_{\varepsilon_\nu}\rangle &= \frac{\mathrm{i}}{\hbar}\langle\Psi_{\varepsilon_\nu}|\left[\overset{\hat\omega}{H}(t),\hat{p}\right]|\Psi_{\varepsilon_\nu}\rangle \\ \frac{\mathrm{d}}{\mathrm{d}t}\langle\Psi_{\varepsilon_\nu}|\hat{q}|\Psi_{\varepsilon_\nu}\rangle &= \frac{\mathrm{i}}{\hbar}\langle\Psi_{\varepsilon_\nu}|\left[\overset{\hat\omega}{H}(t),\hat{q}\right]|\Psi_{\varepsilon_\nu}\rangle\end{aligned} \tag{3.8}$$

we may obtain an equation for the vector $\chi_{(\nu)}(t)$ (see [24, 35]). To do this, we expand the left- and right-hand sides of equation (3.8) in terms of powers of $\hbar^{1/2}$ up to quantities of the order $O(\hbar^{3/2})$. After simple algebraic manipulations we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\chi_{(\nu)} = H_{\mathrm{var}}(t)\chi_{(\nu)} + \tfrac{1}{2}\mathcal{F}_{(\nu)}(t) \tag{3.9}$$

where

$$H_{\mathrm{var}}(t) = \begin{pmatrix} -H_{qp}(t) & -H_{qq}(t) \\ H_{pp}(t) & H_{pq}(t)\end{pmatrix}$$

and

$$\mathcal{F}_{(\nu)}(t) = \left[\begin{pmatrix}-\nabla_q \\ \nabla_p\end{pmatrix}\Box_\sigma\, H(p,q,t)\right]\Bigg|_{\substack{p=p(t)\\ q=q(t)}}. \tag{3.10}$$

Here, $\Box_\sigma$ denotes an operator of the form

$$\hbar\Box_\sigma = \langle\nabla_q, \sigma_{qq}(t)\nabla_q\rangle + 2\langle\nabla_p, \sigma_{pq}(t)\nabla_q\rangle + \langle\nabla_q, \sigma_{qq}(t)\nabla_q\rangle \tag{3.11}$$

where the matrices $\sigma_{qq}(t), \ldots$ are defined by equations (3.3). With equations (3.3), the vectors $\mathcal{F}_{(\nu)}(t)$ can be presented in the form

$$\mathcal{F}_{(\nu)}(t) = \left[\begin{pmatrix}-\nabla_q \\ \nabla_p\end{pmatrix}\sum_{k=1}^{n}(\nu_k+\tfrac{1}{2})\left\{\overset{*}{a}_k(t), H_{\mathrm{var}}(p,q,t)a_k(t)\right\}\right]\Bigg|_{\substack{p=p(t)\\ q=q(t)}}. \tag{3.12}$$

Note that the symplectic vector product in the right-hand side of equation (3.12) is a real quantity.

Equation (3.9) is a non-uniform linear Hamiltonian system with T-periodic coefficients. Its general solution has the form

$$\chi_{(\nu)}(t) = \mathrm{Re}\left[\sum_{k=1}^{n} a_k(t)\left(\frac{1}{2\mathrm{i}}\int_0^t \mathrm{d}\tau\left\{\mathcal{F}_{(\nu)}(\tau), \overset{*}{a}_k(\tau)\right\} + B_{(\nu)k}\right)\right]. \tag{3.13}$$

The integration constants $B_{(\nu)k}$ are found from the condition that the function $\chi_{(\nu)}(t)$ is time-periodic:

$$\chi_{(\nu)}(t+T) = \chi_{(\nu)}(t). \tag{3.14}$$

Let

$$\beta_{(\nu)k}(t) = \frac{1}{2\mathrm{i}}\int_0^t \mathrm{d}\tau\left\{\mathcal{F}_{(\nu)}(\tau), \overset{*}{a}_k(\tau)\right\}. \tag{3.15}$$

Then condition (3.14) means that

$$\mathrm{e}^{+\mathrm{i}\Omega_k T}[\beta_{(\nu)k}(t+T) + B_{(\nu)k}] = \beta_{(\nu)k}(t) + B_{(\nu)k}. \tag{3.16}$$

This equality is fulfilled if

$$B_{(\nu)k} = \frac{\beta_{(\nu)k}(T)}{\mathrm{e}^{-\mathrm{i}\Omega_k T} - 1}. \tag{3.17}$$

Using equation (3.13), we obtain the expression for the Aharonov–Anandan phase (3.7) as

$$\gamma_{\varepsilon_\nu} = \hbar^{-1}\int_0^T \mathrm{d}t \langle p(t), \dot{q}(t)\rangle - \frac{1}{2}\sum_{k=1}^{n}(\nu_k + \tfrac{1}{2})\int_0^T \mathrm{d}t \left\{\dot{\tilde{a}}_k, \overset{*}{\tilde{a}}_k\right\} - \mathrm{Re}\int_0^T \mathrm{d}t \frac{\mathrm{i}}{2}\sum_{l=1}^{n}\{a_0(t), a_l(t)\}$$
$$\times\left[\sum_{k=1}^{n}(\nu_k + \tfrac{1}{2})\int_0^t \mathrm{d}\tau \left\{\overset{*}{a}_k(\tau), \langle\overset{*}{a}_l(\tau), \nabla H_{\mathrm{var}}(\tau)\rangle a_k(\tau)\right\} + B_{(\nu)l}\right] \tag{3.18}$$

where

$$\nabla H_{\mathrm{var}}(\tau) = \left[\begin{pmatrix}\nabla_p \\ \nabla_q\end{pmatrix} H_{\mathrm{var}}(p, q, \tau)\right]\Bigg|_{\substack{p=p(\tau)\\ q=q(\tau)}}. \tag{3.19}$$

The Aharonov–Anandan phase γ_{ε_ν} corresponding to the quasi-energy TCSs Ψ_{ε_ν}, equation (2.31), is thus completely determined by two geometric objects: the closed phase trajectory r_t of the Hamiltonian system (2.1), stable in a linear approximation, and the complex germ $r^n(r_t)$ formed from n linearly independent Floquet solutions to the system in variations, equations (2.2).

4. The Aharonov–Anandan phase γ_{ε_ν} in the adiabatic approximation

Suppose that the evolution of a classical system is defined by a Hamiltonian $H(p, q, R(t))$ depending on time through a set of slowly varying T-periodic functions $R(t) = (R_1(t), \ldots, R_N(t))$. For this case, let us obtain an adiabatic approximation for γ_{ε_ν}. To do this, as follows from equation (3.5), it is necessary to build up adiabatic solutions to equations (2.1), (2.2), and (3.9).

To begin with we construct the adiabatic solution $X(t) = (p(t), q(t))^{\mathrm{T}}$ to the Hamiltonian system (2.1) that would satisfy the condition

$$X(t+T) = X(t). \tag{4.1}$$

To find this solution we use the familiar method as follows. Let us put $t = sT$, where s plays the part of slow dimensionless time. The solution will be found in the form of a formal† asymptotic series in terms of the adiabaticity parameter‡ $1/T$

$$X(t) = \overset{0}{X}(s) + \frac{1}{T}\overset{1}{X}(s) + \mathrm{O}(1/T^2). \tag{4.2}$$

Substitution of equation (4.2) into equation (2.1) results in the relationship

$$\frac{1}{T}\overset{0}{X}{}' = \begin{pmatrix}-H_q(s) \\ H_p(s)\end{pmatrix} + \frac{1}{T}H_{\mathrm{var}}(s)\overset{1}{X} + \mathrm{O}(1/T^2) \tag{4.3}$$

where

$$H_q(s) = H_q(p, q, \tilde{R}(s))\Bigg|_{\substack{p=\overset{0}{p}(s)\\ q=\overset{0}{q}(s)}} \qquad \tilde{R}(s) = R(t)|_{t=sT}$$

† Real dimensionless parameter of adiabatic expansion is $\epsilon = \max_{j=1,N;l=1,n}\left\{\frac{2\pi}{\Omega_l}\cdot\frac{\dot{R}_j}{R_j}\right\}$, where Ω_j is defined in (2.13) (see e.g. [36, 37]).

‡ The problem of verity of two-scale asymptotics has not been widely investigated in the mathematical literature. This question is difficult as there are two limits involved: $T \to \infty$ and $\hbar \to 0$. It can be shown, as $T^{-1} \sim \hbar^{\delta}$, $\delta > 1$, that the expansion into series of $T^{-1} \to 0$ will not impair asymptotics by $\hbar \to 0$ because quasi-energy asymptotics are uniform in t.

and the prime denotes a derivative with respect to s. Then, to the zero order with respect to the adiabatic parameter $1/T$ we have

$$H_p\left(\overset{0}{p}(s), \overset{0}{q}(s), \tilde{R}(s)\right) = 0 \tag{4.4}$$

$$H_q\left(\overset{0}{p}(s), \overset{0}{q}(s), \tilde{R}(s)\right) = 0 \tag{4.5}$$

and, hence, the vector $\overset{0}{X}(s) = (\overset{0}{p}(s), \overset{0}{q}(s))^T$ specifies the rest-point for the function $H(p, q, \tilde{R}(s))$ at every fixed value of s. Thus, we have

$$\overset{0}{p}(s) = P_0(\tilde{R}(s)) = P_0(R(t)) \qquad \overset{0}{q}(s) = Q_0(\tilde{R}(s)) = Q_0(R(t)). \tag{4.6}$$

Obviously, to the zero approximation, the periodicity condition (4.1) is fulfilled.

To the first approximation, according to equation (4.3), we obtain the algebraic system of equations

$$H_{\mathrm{var}}(s)\overset{1}{X} = \overset{0}{X}{}'. \tag{4.7}$$

If the assumption is made that the rest-point given by equations (4.6) is stable (for every s) in the linear approximation, then there exists a set of linearly independent eigenvectors $f_k(s) = a_k(\tilde{R}(s))$, $k = \overline{1, n}$, of the matrix $H_{\mathrm{var}}(s)$:

$$H_{\mathrm{var}}(s) f_k = \mathrm{i}\overset{0}{\Omega}_k\left(\tilde{R}(s)\right) f_k \qquad \mathrm{Im}\,\overset{0}{\Omega}_k = 0 \tag{4.8}$$

such that

$$\{f_k, f_l\} = 0 \qquad \left\{f_k, \overset{*}{f}_l\right\} = 2\mathrm{i}\delta_{kl} \qquad k, l = \overline{1, n}. \tag{4.9}$$

According to Maslov's terminology [21], this set of vectors specifies the simplest complex germ on a zero-dimensional Lagrangian manifold. The solution of the system of equations (4.7) with equation (4.1) has the form

$$\overset{1}{X}(s) = \begin{pmatrix} \overset{1}{p}(s) \\ \overset{1}{q}(s) \end{pmatrix} = \mathrm{Re}\left[\sum_{k=1}^{n} \frac{f_k(s)}{\overset{0}{\Omega}_k\left(\tilde{R}(s)\right)} \left\{\overset{*}{f}_k(s), \overset{0}{X}{}'(s)\right\}\right]. \tag{4.10}$$

Thus, for the case where formulae (4.6) and (4.10) hold, the function (4.2) is the sought-for adiabatic solution to the Hamiltonian system (2.1).

Let us now construct adiabatic solutions to the system in variations (2.2) that would satisfy the conditions (2.3) and (2.4). Expanding the matrix $H_{\mathrm{var}}(t)$ in the neighbourhood of the stationary point specified by equations (4.6) we have

$$H_{\mathrm{var}}(t) = H_{\mathrm{var}}(s) + \frac{1}{T}\left\langle \nabla H_{\mathrm{var}}(s), \overset{1}{X}(s)\right\rangle + \mathrm{O}(1/T^2) \tag{4.11}$$

where $\nabla H_{\mathrm{var}}(s)$ is given by equation (3.19). We look for the solution as an expansion in terms of the eigenvectors $f_l(s)$, $\overset{*}{f}_l(s)$:

$$a_k(t) = \sum_{l=1}^{n}\left[A_{lk}(s,\theta) f_l(s) + \tilde{A}_{lk}(s,\theta)\overset{*}{f}_l\right] \tag{4.12}$$

with coefficients

$$A_{lk}(s,\theta) = \overset{0}{A}_{lk}(s,\theta) + \frac{1}{T}\overset{1}{A}_{lk}(s,\theta) + O(1/T^2)$$
$$\tilde{A}_{lk}(s,\theta) = \overset{0}{\tilde{A}}_{lk}(s,\theta) + \frac{1}{T}\overset{1}{\tilde{A}}_{lk}(s,\theta) + O(1/T^2). \tag{4.13}$$

Here, $\theta = (T\Phi_1(s), \ldots, T\Phi_n(s))$ is a set of 'rapid' variables where the real functions $\Phi_k(s)$, $k = \overline{1,n}$, are unknowns independent of T. Obviously, the time derivative in this case is given by $\partial_t = \frac{1}{T}\partial_s + \Phi'(s)\partial_\theta$. Substituting equations (4.12) and (4.13) into equation (2.2) and equating the summands with the same power of $1/T$, we obtain, to the zero order,

$$\sum_{j=1}^{n}\Phi'_j\frac{\partial \overset{0}{A}_{lk}}{\partial\theta_j} = \mathrm{i}\,\overset{0}{\Omega}_l\,\overset{0}{A}_{lk} \qquad \sum_{j=1}^{n}\Phi'_j\frac{\partial \overset{0}{\tilde{A}}_{lk}}{\partial\theta_j} = \mathrm{i}\,\overset{0}{\Omega}_l\,\overset{0}{\tilde{A}}_{lk}. \tag{4.14}$$

Integration of equations (4.14) gives

$$\overset{0}{A}_{lk}(s,\theta) = \sum_{j=1}^{n}\overset{0}{A}_{lkj}(s)\mathrm{e}^{\mathrm{i}\theta_j\overset{0}{\Omega}_l/\Phi'_j} \qquad \overset{0}{\tilde{A}}_{lk}(s,\theta) = \sum_{j=1}^{n}\overset{0}{\tilde{A}}_{lkj}(s)\mathrm{e}^{-\mathrm{i}\theta_j\overset{0}{\Omega}_l/\Phi'_j} \tag{4.15}$$

where $\overset{0}{A}_{lkj}(s)$, $\overset{0}{\tilde{A}}_{lkj}(s)$ are the integration constants. The additional requirement that the functions (4.15) be 2π-periodic with respect to all rapid variables θ_j, $j = \overline{1,n}$ makes it possible to find the functions

$$\Phi_j(s) = \int_0^s \overset{0}{\Omega}_j(\tilde{R}(s))\,\mathrm{d}s \tag{4.16}$$

and sets the following limitation for the choice of the integration constants:

$$\overset{0}{A}_{lkj}(s) = \overset{0}{C}_{lk}(s)\delta_{lj} \qquad \overset{0}{\tilde{A}}_{lkj} = \overset{0}{\tilde{C}}_{lk}(s)\delta_{lj}. \tag{4.17}$$

From the quasi-periodicity condition (2.3) for the solutions (4.12) and from equations (2.4), it, in turn, follows that

$$\overset{0}{C}_{lk}(s) = \delta_{lk}\mathrm{e}^{-\mathrm{i}\mathcal{N}_l(s)} \qquad \overset{0}{\tilde{C}}_{lk}(s) = 0. \tag{4.18}$$

As a result, the zero approximation to the functions $a_k(t)$ is given by

$$\overset{0}{a}_k(t) = f_k(s)\mathrm{e}^{\mathrm{i}(\theta_k - \mathcal{N}_k(s))} \qquad \theta_k = T\int_0^s \overset{0}{\Omega}_k(\tilde{R}(s))\,\mathrm{d}s \tag{4.19}$$

where the real functions $\mathcal{N}_k(s)$ remain unknown.

Then, to a first-order adiabatic approximation, we obtain the set of equations for the functions $\overset{1}{A}_{lk}$, $\overset{1}{\tilde{A}}_{lk}$:

$$\sum_{j=1}^{n}\overset{0}{\Omega}_j\frac{\partial \overset{1}{A}_{lk}}{\partial\theta_j} - \mathrm{i}\,\overset{0}{\Omega}_l\,\overset{1}{A}_{lk} - \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(\theta_k-\mathcal{N}_k)}\left(2\delta_{lk}\mathcal{N}'_k - \left\{\overset{*}{f}_l, \frac{\mathrm{D}}{\mathrm{d}s}f_k\right\}\right) = 0$$
$$\sum_{j=1}^{n}\overset{0}{\Omega}_j\frac{\partial \overset{1}{A}_{lk}}{\partial\theta_j} + \mathrm{i}\,\overset{0}{\Omega}_l\,\overset{1}{\tilde{A}}_{lk} - \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(\theta_k-\mathcal{N}_k)}\left\{f_l, \frac{\mathrm{D}}{\mathrm{d}s}f_k\right\} = 0 \tag{4.20}$$

where

$$\frac{\mathrm{D}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s} - \left\langle \nabla H_{\mathrm{var}}(s), \overset{1}{X}(s)\right\rangle. \tag{4.21}$$

As in the above case, we see the solutions to equations (4.20) complemented with the condition for the 2π-periodicity with respect to the variables θ_j, $j = \overline{1, n}$, and relationships (2.3) and (2.4). Simple manipulation results in

$$\mathcal{N}_k(s) = \frac{1}{2}\int_0^s \mathrm{d}s \left\{ \overset{*}{f}_k, \frac{\mathrm{D}}{\mathrm{d}s} f_k \right\} \tag{4.22}$$

$$\overset{1}{A}_{lk}(s,\theta) = \mathrm{e}^{\mathrm{i}(\theta_k - \mathcal{N}_k(s))} \left[\overset{1}{C}_k(s)\delta_{lk} + \frac{1-\delta_{lk}}{2\left(\overset{0}{\Omega}_l - \overset{0}{\Omega}_k\right)} \left\{ \overset{*}{f}_l, \frac{\mathrm{D}}{\mathrm{d}s} f_k \right\} \right] \tag{4.23}$$

$$\overset{1}{\tilde{A}}_{lk}(s,\theta) = \mathrm{e}^{\mathrm{i}(\theta_k - \mathcal{N}_k(s))} \frac{1}{2\left(\overset{0}{\Omega}_l + \overset{0}{\Omega}_k\right)} \left\{ f_l, \frac{\mathrm{D}}{\mathrm{d}s} f_k \right\}. \tag{4.24}$$

The functions $\overset{1}{C}_k(s)$ entering onto equation (4.23) are determined from subsequent approximations; moreover, they satisfy the condition $\overset{1}{C}_k(s+1) = \overset{1}{C}_k(s)$. It should be stressed that relationships (4.23) and (4.24) have been obtained with the supposition that there are no resonance correlations between the frequencies, i.e. $\sum_{j=1}^n m_j \overset{0}{\Omega}_j(\tilde{R}(s)) \neq 0$, $m_j \in \mathbb{Z}$.

Substitution of equations (4.23) and (4.24) into equation (4.12), and consideration of equation (4.15) results in

$$a_k(t) = \mathrm{e}^{\mathrm{i}(\theta_k - \mathcal{N}_k(s))} \sum_{l=1}^n \left(\left[\left(1 + \frac{1}{T}\overset{1}{C}_k(s)\right)\delta_{lk} + \frac{1-\delta_{lk}}{2T\left(\overset{0}{\Omega}_l - \overset{0}{\Omega}_k\right)} \left\{ \overset{*}{f}_l, \frac{\mathrm{D}}{\mathrm{d}s} f_k \right\} \right] f_l \right.$$
$$\left. + \frac{1}{2T\left(\overset{0}{\Omega}_l + \overset{0}{\Omega}_k\right)} \left\{ f_l, \frac{\mathrm{D}}{\mathrm{d}s} f_k \right\} \overset{*}{f}_l \right) + \mathrm{O}(1/T^2). \tag{4.25}$$

Comparing equation (4.25) with the quasi-periodicity condition (2.3) we obtain

$$\Omega_k = \int_0^1 \overset{0}{\Omega}_k(\tilde{R}(s))\,\mathrm{d}s - \frac{1}{2T}\int_0^1 \left\{ \overset{*}{f}_k, \frac{\mathrm{D}}{\mathrm{d}s} f_k \right\} \mathrm{d}s + \mathrm{O}(1/T^2). \tag{4.26}$$

The adiabatic solution to equation (3.9) is built up using the same scheme. As a result, the sought-for $(\mathrm{mod}\, T^{-1})$ solution that satisfies the T-periodicity condition (3.14) is presented as

$$\chi_{(\nu)}(t) = \frac{1}{2}\mathrm{Re}\left[\sum_{k=1}^n \overset{*}{f}_k(s) \frac{1}{\overset{0}{\Omega}_k(\tilde{R}(s))} \{\mathcal{F}_{(\nu)}(s), f_k(s)\} \right] + \mathrm{O}(1/T) \tag{4.27}$$

where $\mathcal{F}_{(\nu)}(s)$ is the zero approximation for the vector function $\mathcal{F}_{(\nu)}(t)$ given by equation (3.12).

Substituting equations (4.19), (4.26) and (4.27) into equation (3.5), we obtain

$$\gamma_{\varepsilon_\nu} = \beta_\nu(C) + \mathrm{O}(1/T) \tag{4.28}$$

where

$$\beta_\nu(C) = \frac{1}{\hbar}\oint_C \left\langle P_0(R), \frac{\partial Q_0(R)}{\partial R_i} \right\rangle \mathrm{d}R_i + \sum_{k=1}^n (\nu_k + \tfrac{1}{2}) \oint_C \left\{ a_k^*(R), \mathcal{T}_k^{(i)}(R) \right\} \mathrm{d}R_i \tag{4.29}$$

and

$$\mathcal{T}_k^{(i)}(R) = \frac{1}{2}\frac{\partial a_k(R)}{\partial R_i} - \frac{1}{2}\sum_{l=1}^{n}\frac{1}{\overset{0}{\Omega}_k(R)} \times \left[\mathrm{Re}\left(\langle\nabla H_{\mathrm{var}}(R), a_l(R)\rangle\left\{\overset{*}{a}_l(R), \frac{\partial \overset{0}{X}(R)}{\partial R_i}\right\}\right)\right] a_k(R). \tag{4.30}$$

Here, C is the closed loop drawn by the end of the vector $R(t)$, $t \in [0, T]$ in the space of parameters $(R_1, \ldots, R_N)$.

As follows from the results of [19]†, the expression for $\beta_\nu(C)$ given by formulae (4.29) and (4.30) coincides with the quasi-classical expression for the Berry phase associated with the adiabatic motion of a zero-dimensional Lagrangian manifold with a complex germ.

5. Examples

As the first example illustrating the general scheme described in section 3, let us consider the one-dimensional sinusoidally forced harmonic oscillator [31, 38, 39]. In this case, the Hamiltonian of the system has the form

$$\hat{H}(t) = \tfrac{1}{2}\hat{p}^2 + \tfrac{1}{2}\omega^2\hat{q}^2 - qF\sin\omega_0 t \tag{5.1}$$

where ω, ω_0, and F are constants. Thus, we have $\hat{H}(t+T) = \hat{H}(t)$ with $T = 2\pi/\omega_0$. The Hamiltonian (5.1) conforms with the classical Hamiltonian

$$H(t) = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2 q^2 - qF\sin\omega_0 t. \tag{5.2}$$

The classical set of equations (2.1) for this Hamiltonian admits a T-periodic solution describing an ellipse:

$$p = \frac{F\omega_0}{\omega^2 - \omega_0^2}\cos\omega_0 t \qquad q = \frac{F}{\omega^2 - \omega_0^2}\sin\omega_0 t \qquad \omega \neq \omega_0. \tag{5.3}$$

In view of the fact that the matrix $H_{\mathrm{var}} = $ constant, the problem of constructing the Floquet solutions (2.3), normalized by the conditions (2.4), is reduced to solving the spectral problem for the matrix H_{var}. Eventually, we find

$$a(t) = \mathrm{e}^{\mathrm{i}\omega t} f_0 \qquad f_0 = \begin{pmatrix} \sqrt{\omega} \\ -\mathrm{i}/\sqrt{\omega} \end{pmatrix}. \tag{5.4}$$

But, according to equation (3.6) we have

$$\tilde{a}(t) = f_0 = \text{constant}. \tag{5.5}$$

From equations (2.39) and (3.18) it follows that, in this case, the phase γ_{ε_ν} is independent of ν and is given by

$$\hbar\gamma_{r} = \int_0^T \mathrm{d}t\, p(t)\dot{q}(t) = \frac{F^2\omega_0\pi}{(\omega^2 - \omega_0^2)^2}. \tag{5.6}$$

† [19] contains the following errors:

(1) There are bound to be $-\nu_+$ instead of $+\nu_+$ in (6.9), (6.12) and (6.13).

(2) Matrix $\vec{\sigma}$ was omitted in (5.27) which must have the form

$$T_\zeta^{(i)} = \mathrm{i}\frac{\partial \nu_\zeta(R)}{\partial R_i} + \frac{e_0}{2mc}\left\langle\vec{\sigma}, \left[\frac{1}{c}\frac{\partial \vec{x}_0(R)}{\partial R_i} X\vec{E}(R) - \sum_{k=1}^{3}\frac{1}{\Omega_k(R)}\mathrm{Re}\left(\langle\nabla\vec{H}(R), a_k(R)\rangle\left\{\overset{*}{a}_k(R), \frac{\partial \overset{0}{X}(R)}{\partial R_i}\right\}\right)\right]\right\rangle\nu_\zeta(R).$$

Thus, it has a simple quasi-classical sense: the area covered by the radius vector of a point moving over the phase plane.

Now let us consider the motion of a relativistic spinless particle describing by the Klein–Gordon equation

$$[\hat{P}_0^2 - c^2\hat{P}^2 - m_0^2c^4]\Psi = 0 \qquad \hat{P}_0 = \mathrm{i}\hbar\partial_t - eA_0 \qquad \hat{P} = -\mathrm{i}\hbar\nabla - \frac{e}{c}A \tag{5.7}$$

in the Redmond field

$$A_0 = 0 \qquad A = \left(-\frac{cE_0}{\omega}\sin\omega\xi + \frac{H}{2}y, \frac{cE_0 g}{\omega}\cos\omega\xi - \frac{H}{2}x, 0\right) \tag{5.8}$$

where $\xi = t - z/c$, g defines right ($g = 1$) and left ($g = -1$) circular polarizations and E_0 is an amplitude of the electric-field strength. The problem of (1.3) and (1.8) in the field (5.8) can be reduced, for the equation (5.7), to the problem considered before in the special coordinate system.

Let us transform to coordinates of the zero plane [40] in equation (5.7):

$$u_0 = t - z/c \qquad u_1 = x \qquad u_2 = y \qquad u_3 = t + z/c. \tag{5.9}$$

Equations (5.7) and (5.8) take the following form in coordinates of the zero plane (5.9):

$$\begin{gathered}\{4\hat{p}_0\hat{p}_3 - c^2\hat{\mathcal{P}}_2^2 - c^2\hat{\mathcal{P}}_2^2 - m_0^2c^4\}\Psi = 0 \\ \hat{\mathcal{P}}_k = \hat{p}_k - \frac{e}{c}A_k \qquad \hat{p}_k = -\mathrm{i}\hbar\partial_{u_k} \qquad k = \overline{0,3}\end{gathered} \tag{5.10}$$

and the scalar product is

$$\langle\Psi_1 \mid \Psi_2\rangle = \int \mathrm{d}^3u[\Psi_1^*\hat{p}_3\Psi_2 + (\hat{p}_3\Psi_1)^*\Psi_2]. \tag{5.11}$$

A solution of equation (5.10) will be found in the form

$$\Psi(\boldsymbol{u}, u_0, \hbar) = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{\mathrm{i}}{\hbar}\lambda u_3\right)\tilde{\Psi}(\boldsymbol{u}_\perp, u_0, \hbar)) \qquad \boldsymbol{u}_\perp = (u_1, u_2). \tag{5.12}$$

Thus, the function $\tilde{\Psi}$ satisfies the Schrödinger equation

$$\left\{-\mathrm{i}\hbar\partial_{u_0} + \hat{\tilde{\mathcal{H}}}(u_0)\right\}\tilde{\Psi} = 0 \tag{5.13}$$

where

$$\hat{\tilde{\mathcal{H}}}(u_0) = \frac{c^2}{4\lambda}[\hat{\mathcal{P}}_1^1 + \hat{\mathcal{P}}_2^2 + m_0^2c^2]. \tag{5.14}$$

A scalar product for the functions $\tilde{\Psi}$ has the form

$$\langle\tilde{\Psi}_1 \mid \tilde{\Psi}_2\rangle = 2\lambda\int \mathrm{d}^2u\tilde{\Psi}_1^*\tilde{\Psi}_2. \tag{5.15}$$

Hence, for the function $\tilde{\Psi}$ one obtains the Schrödinger equation with the periodic Hamiltonian (5.14), where u_0 plays the part of time.

The Hamiltonian system corresponding to the problem of (5.13) and (1.2) has the form

$$\begin{gathered}\dot{\boldsymbol{u}}_\perp = \frac{c^2}{2\lambda}\boldsymbol{\mathcal{P}}_\perp \qquad \dot{\boldsymbol{p}}_\perp = -\frac{\mathrm{i}\omega_0}{2}\sigma_2\boldsymbol{\mathcal{P}}_\perp \\ \boldsymbol{\mathcal{P}}_\perp = \boldsymbol{p}_\perp - \mathrm{i}\frac{eH}{2c}\sigma_2\boldsymbol{u}_\perp - \frac{e}{c}\boldsymbol{\mathcal{A}}_\perp \qquad \boldsymbol{\mathcal{A}}_\perp = \frac{cE_0 g}{\omega}\exp(-\mathrm{i}g\sigma_2\omega u_0)\begin{pmatrix}0\\1\end{pmatrix} \qquad \omega_0 = \frac{ecH}{2\lambda}\end{gathered} \tag{5.16}$$

$$u_\perp(t+T) = u_\perp(t) \qquad p_\perp(t+T) = p_\perp(t) \qquad p_\perp = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \qquad u_\perp = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

Excluding $p_\perp$ from system (5.16) one can write

$$\ddot{u}_\perp + \mathrm{i}\omega_0\sigma_2\dot{u}_\perp = -\frac{ec}{2\lambda}\dot{\mathcal{A}}_\perp.$$

After integrating this equation one has

$$\dot{u}_\perp + \mathrm{i}\omega_0\sigma_2 u_\perp = -\frac{ec}{2\lambda}\mathcal{A}_\perp + \alpha_\perp \qquad \alpha_\perp = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \text{constant}. \tag{5.17}$$

In the resonanceless conditions, e.g. $\omega_0 \neq \omega$,

$$u_\perp = \mathrm{e}^{-\mathrm{i}\sigma_2\omega_0 u_0} v_\perp - \frac{\mathrm{i}}{\omega_0}\sigma_2\alpha_\perp - \frac{ec}{2\lambda}\mathrm{e}^{-\mathrm{i}\sigma_2\omega_0 u_0}\int_0^{u_0}\mathrm{e}^{\mathrm{i}\sigma_2\omega_0\tau}\mathcal{A}_\perp(\tau)\,\mathrm{d}\tau \tag{5.18}$$

where $v_\perp$ = constant, one obtains the periodic solution in the form

$$u_\perp = -\frac{\mathrm{i}}{\omega_0}\sigma_2\alpha_\perp + \frac{\mathrm{i}}{\omega_0 - g\omega}\frac{ec}{2\lambda}\sigma_2\mathcal{A}_\perp. \tag{5.19}$$

From (5.16) and (5.19) one finds

$$p_\perp = \frac{2\lambda}{c^2}\dot{u}_\perp + \frac{e}{c}\mathcal{A}_\perp - \frac{eH}{2c}\mathrm{i}\sigma_2 u_\perp + \frac{e}{2c(\omega_0 - g\omega)}\mathcal{A}_\perp. \tag{5.20}$$

The system in variations responding to the Hamiltonian (5.14) can be written as

$$\dot{W} = \mathrm{i}\frac{\omega_0}{2}\sigma_2 W + \frac{\lambda\omega_0^2}{2c^2}Z \qquad \dot{Z} = \frac{c^2}{2\lambda}W - \mathrm{i}\frac{\omega_0}{2}\sigma_2 Z. \tag{5.21}$$

Thus,

$$\ddot{Z} + \mathrm{i}\omega_0\sigma_2\dot{Z} = 0. \tag{5.22}$$

Denote eigenvectors of the matrix σ_2 as f_ζ:

$$\sigma_2 f_\zeta = \zeta f_\zeta \qquad \langle f_\zeta, f_{\zeta'}\rangle = \delta_{\zeta,\zeta'} \qquad \zeta, \zeta' = \pm 1 \qquad f_\zeta = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{i}\zeta \\ 1 \end{pmatrix}. \tag{5.23}$$

So,

$$a_k(u_0) = N_k\begin{pmatrix} W^k \\ Z^k \end{pmatrix} \qquad k = 1, 2 \tag{5.24}$$

where

$$\begin{aligned} Z^1 &= \mathrm{e}^{-\mathrm{i}\zeta\omega_0 u_0} f_\zeta \qquad W^1 = -\mathrm{i}\zeta\frac{\lambda\omega_0}{c^2}\mathrm{e}^{-\mathrm{i}\zeta\omega_0 u_0} f_\zeta \\ Z^2 &= f_{-\zeta} \qquad W^2 = -\mathrm{i}\zeta\frac{\lambda\omega_0}{c^2} f_{-\zeta}. \end{aligned} \tag{5.25}$$

From the normalized condition $\{a_k, a_l^*\} = 2\mathrm{i}\delta_{k,l}$ one finds the constants N_k and ζ and, finally,

$$N_1^2 = N_2^2 = \frac{c^2}{\lambda\omega_0^2} \qquad \zeta = -1 \tag{5.26}$$

and for the quasi-energy spectrum one finds

$$\varepsilon_\lambda = \frac{1}{T}\int_0^T (\tilde{\mathcal{H}}(t) - \langle p_\perp, \dot{u}_\perp\rangle)\,\mathrm{d}t + \hbar\omega_0(\nu_1 + \tfrac{1}{2}). \tag{5.27}$$

Now let us calculate the integrals:

$$\int_0^T \langle p_\perp, \dot{u}_\perp \rangle \, dt = \frac{ge^2c^2\omega_0 E_0^2}{4\omega\lambda(\omega_0 - g\omega)^2} T \qquad \int_0^T \tilde{\mathcal{H}}(t)\, dt = \left(\frac{e^2c^2E_0^2}{4\lambda(\omega_0 - g\omega)^2} + \frac{m_0^2c^4}{4\lambda} \right) T.$$

Thus, for the quasi-energy spectrum and for the Aharonov–Anandan phase one obtains

$$\begin{aligned} \varepsilon_{\lambda,\nu} &= \frac{ge^2c^2E_0^2}{4\omega\lambda(\omega_0 - g\omega)} + \frac{m_0^2c^4}{4\lambda} + \hbar\omega_0(\nu + \tfrac{1}{2}) \\ \hbar\gamma_\nu &= \int_0^T \langle p_\perp, \dot{u}_\perp \rangle \, dt = \frac{ge^2c^2\omega_0 E_0^2 T}{4\omega\lambda(\omega_0 - g\omega)^2}. \end{aligned} \tag{5.28}$$

Thus, in the field (5.8) the semiclassical limit of the Aharonov–Anandan phase is, essentially, the symplectic area averaged over the motion of $p_\perp$, $u_\perp$ (see also [41]).

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Appendix A.

Firstly we use equation (2.37) to find the constants $\overset{1}{C}_{\nu\nu'}$. To do this, it is convenient to rewrite equation (2.28) in the form

$$\varphi_\nu^{(1)} = \sum_{|\nu'|=0}^{\infty} \left(\beta_{\nu\nu'}^{(1)}(t) + \overset{1}{C}_{\nu\nu'} \right) |\nu', t\rangle \tag{A.1}$$

where

$$\beta_{\nu\nu'}^{(1)}(t) = -\frac{\mathrm{i}}{3!\hbar^{3/2}} \int_0^t d\tau \langle \nu', \tau | \hat{\delta}^3 H(\tau) | \nu, \tau \rangle. \tag{A.2}$$

In view of the fact that the set of functions $|\nu, t\rangle$ is orthogonal and complete, we conclude, based on equation (2.37), that

$$\mathrm{e}^{-\mathrm{i}\beta_{\nu'}T} \{ \beta_{\nu\nu'}^{(1)}(t+T) + \overset{1}{C}_{\nu\nu'} \} = \mathrm{e}^{-\mathrm{i}\beta_\nu T} \left\{ \beta_{\nu\nu'}^{(1)}(t) + \overset{1}{C}_{\nu\nu'} \right\}. \tag{A.3}$$

It can readily be seen that with a properly chosen constant $\overset{1}{C}_{\nu\nu'}$, relationship (A.3) is fulfilled for any value of t. To do this, it is sufficient to differentiate equation (A.3) with respect to t and be convinced that this will give an identity. Then, setting $t = 0$, we obtain from equation (A.3),

$$\overset{1}{C}_{\nu\nu'} = \frac{\beta_{\nu\nu'}^{(1)}(T)(1 - \delta_{\nu\nu'})}{\mathrm{e}^{-\mathrm{i}(\beta_\nu - \beta_{\nu'})T} - 1} + \overset{1}{C}_\nu \delta_{\nu\nu'}. \tag{A.4}$$

Note that for $\nu = \nu'$, equation (A.3) becomes an identity since, in this case, the coefficients $\beta_{\nu\nu}^{(1)}(t)$ are zero in view of equations (2.21) and (2.23).

The constants $\overset{2}{C}_{\nu\nu'}$ and $\varepsilon_\nu^{(2)}$ can be found in a similar way from equation (2.38). Let us put

$$\begin{aligned}\beta_{\nu\nu'}^{(2)}(t) &= -\frac{\mathrm{i}}{4!\hbar^2}\int_0^t \mathrm{d}\tau \langle \nu', \tau | \hat{\delta}^{(4)} H(\tau) | \nu, \tau\rangle \\ \alpha_{\nu\nu'}(t) &= -\frac{\mathrm{i}}{3!\hbar^{3/2}}\int_0^t \mathrm{d}\tau \langle \nu', \tau | \hat{\delta}^{(3)} H(\tau) | \varphi_\nu^{(1)}\rangle.\end{aligned} \tag{A.5}$$

Then, function (2.29) can be represented in the form

$$\varphi_\nu^{(2)} = \sum_{|\nu'|=0}^{\infty} \left\{\alpha_{\nu\nu'}(t) + \beta_{\nu\nu'}^{(2)}(t) + \overset{2}{C}_{\nu\nu'}\right\} |\nu', t\rangle. \tag{A.6}$$

Substituting equation (A.6) into equation (2.35), we obtain the condition

$$\begin{aligned}&\mathrm{e}^{-\mathrm{i}\langle\beta_{\nu'}\rangle T}\{\alpha_{\nu\nu'}(t+T) + \beta_{\nu\nu'}^{(2)}(t+T) + \overset{2}{C}_{\nu\nu'}\} \\ &\qquad = \mathrm{e}^{-\mathrm{i}\langle\beta_\nu\rangle T}\{\alpha_{\nu\nu'}(t) + \beta_{\nu\nu'}^{(2)}(t) + \overset{2}{C}_{\nu\nu'} - \mathrm{i}\varepsilon_\nu^{(2)} T \delta_{\nu\nu'}\}.\end{aligned} \tag{A.7}$$

By analogy with equation (A.3), from equation (A.7) it follows that

$$\overset{2}{C}_{\nu\nu'} = \frac{[\alpha_{\nu\nu'}(T) + \beta_{\nu\nu'}^{(2)}(T)](1-\delta_{\nu\nu'})}{\mathrm{e}^{-\mathrm{i}\langle\beta_\nu - \beta_{\nu'}\rangle T} - 1} + \overset{2}{C}_\nu \delta_{\nu\nu'} \tag{A.8}$$

$$\varepsilon_\nu^{(2)} = \frac{\mathrm{i}}{T}\{\alpha_{\nu\nu}(T) + \beta_{\nu\nu}^{(2)}(T)\}. \tag{A.9}$$

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